A REPRESENTATION THEORETIC APPROACH TO KOHNEN'S PLUS SPACE OF MODULAR FORMS OF HALF INTEGRAL WEIGHT

HUNG YEAN LOKE AND GORDAN SAVIN

Abstract. In this paper, we define a notion of pseudo-spherical type for the two fold central extension of $\mathrm{SL}_2(\mathbb{Q}_2)$. We relate this definition to some results in classical modular forms of half integral weights.

1. Introduction

Let $\mathbb{Q}$ be the field of rational numbers. For every place v of $\mathbb{Q}$ let $\mathbb{Q}_v$ denote the corresponding local field. Then $\mathbb{Q}_v = \mathbb{R}$ or $\mathbb{Q}_p$ for a prime p. The group $\mathrm{SL}_2(\mathbb{Q}_v)$ has a non-trivial two-fold central extension

$$1 \to \mu_2 \to G(\mathbb{Q}_v) \to \mathrm{SL}_2(\mathbb{Q}_v) \to 1 \tag{1}$$

where $\mu_2 = \{\pm 1\}$. Recall that an irreducible representation of $G(\mathbb{Q}_v)$ is called genuine if the central subgroup μ_2 acts faithfully on it. Gelbart's book [G2] contains a basic theory of genuine representations of $G(\mathbb{R})$ and $G(\mathbb{Q}_p)$ for $p \neq 2$. Our intent is to develop a theory in the case of $G(\mathbb{Q}_2)$. The main difference between $G(\mathbb{Q}_2)$ and $G(\mathbb{Q}_p)$ for $p \neq 2$ lies in the fact that the central extension splits over $\mathrm{SL}_2(\mathbb{Z}_p)$ for $p \neq 2$. In particular, we have a subgroup $K_p \subseteq G(\mathbb{Q}_p)$ isomorphic to $\mathrm{SL}_2(\mathbb{Z}_p)$ under the natural projection from $G(\mathbb{Q}_p)$ to $\mathrm{SL}_2(\mathbb{Q}_p)$ for every $p \neq 2$. A genuine representation π of $G(\mathbb{Q}_p)$ is called *unramified* if it contains a non-zero K_p-fixed vector.

Assume now that $p = 2$. Let K denote the full inverse image of $\mathrm{SL}_2(\mathbb{Z}_2)$ in $G(\mathbb{Q}_2)$. In this case the central extension splits over a smaller subgroup. More precisely, we have a subgroup $K_1(4) \subseteq K$ isomorphic to the subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$ given by the following congruence:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod 4$$

In this paper we completely describe genuine irreducible representations of $G(\mathbb{Q}_2)$ containing non-zero $K_1(4)$-fixed vectors. More precisely, in Section 3, we describe a Hecke algebra $H(\gamma)$ which captures the structure of all representations generated by $K_1(4)$-fixed vectors and with a fixed central character γ. In Section 4 we show that $H(\gamma)$ is isomorphic to the Iwahori-Matsumoto Hecke algebra for $\mathrm{PGL}_2(\mathbb{Q}_2)$. In this way we get a correspondence between (some) representations of $G(\mathbb{Q}_2)$ and representations of $\mathrm{PGL}_2(\mathbb{Q}_2)$. We call this correspondence a local Shimura correspondence.

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In Section 5 we show that the compact group K has exactly two irreducible genuine representations, with the fixed central character γ, containing non-zero $K_1(4)$-fixed vectors. These representations are denoted by $V(2)$ and $V(-1)$ and have dimensions 2 and 4, respectively. We show that a representation π of $G(\mathbb{Q}_2)$ has $V(2)$ as a K-type if and only if it corresponds to an unramified representation of $\mathrm{PGL}_2(\mathbb{Q}_2)$, by the local Shimura correspondence. Thus, it is natural to define unramified representations of $G(\mathbb{Q}_2)$ to be those that contain $V(2)$ as a K-type, and we call $V(2)$ a pseudo-spherical type.

We should point out that the center of $G(\mathbb{Q}_2)$ is a cyclic group of order 4. Thus, we have two different genuine central characters γ and two classes of unramified representations. This is analogous to the case of the real group $G(\mathbb{R})$, where the weights $-1/2$ and $1/2$ are called pseudo-spherical types.

We then apply our local results in a global setting in Section 8. Let $\mathbb{A}$ be the ring of adeles and let $G(\mathbb{A})$ be the two-fold cover of $\mathrm{SL}_2(\mathbb{A})$. Let $r > 1$ be an odd integer. Let $\pi = \otimes \pi_v$ be a genuine cuspidal automorphic representation such that

- π_∞ is a holomorphic discrete series representation with the lowest weight $r/2$.
- π_p is unramified for all $p \neq 2$.
- π_2 contains a non-zero $K_1(4)$-fixed vector.

Every such π corresponds to a Hecke eigenspace in $S_{r/2}(\Gamma_0(4))$, the space of cuspidal modular forms of weight $r/2$. Roughly speaking, a function $f = \otimes f_v$ in π such that f_∞ is a lowest weight vector in π_∞, f_p is K_p-fixed and f_2 is $K_1(4)$-fixed, gives naturally a modular form in $S_{r/2}(\Gamma_0(4))$. Since the space of $K_1(4)$-fixed vectors in π_2 is 2-dimensional, unless π_2 is a Steinberg representation, the cuspidal automorphic representation π gives rise to a two-dimensional Hecke eigenspace in $S_{r/2}(\Gamma_0(4))$. We can pick a line in this subspace by taking f_2 to be in the K-type isomorphic to $V(2)$. In this way we get a representation-theoretic description of Kohnen's plus space $S^+_{r/2}(\Gamma_0(4))$ [Ko]. Finally, we show that the global Shimura correspondence is compatible with our local Shimura correspondence at the place $p = 2$.

2. Double cover of $\mathrm{SL}_2(\mathbb{Q}_v)$

We will describe the double cover $G(\mathbb{Q}_v)$ in (1). If we fix a section $\mathbf{s} : \mathrm{SL}_2(\mathbb{Q}_v) \to G(\mathbb{Q}_v)$ then $G(\mathbb{Q}_v)$ can be identified with the set $\mathrm{SL}_2(\mathbb{Q}_v) \times \mu_2$, and the group law on $G(\mathbb{Q}_v)$ is given by

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \sigma_v(g_1, g_2))$$

where $\sigma_v(g_1, g_2)$ is a cocycle which depends on $\mathbf{s}$. Following [G2] we make the following choice of the cocycle σ_v. Let $(,)_v$ be the Hilbert symbol over $\mathbb{Q}_v$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_v)$ we define

$$x(g) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0; \end{cases} \quad \text{and} \quad s(g) = \begin{cases} (c, d)_v & \text{if } v \text{ is a finite prime, } cd \neq 0 \\ & \text{and } \mathrm{ord}(c) \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$\sigma_v(g_1, g_2) = (x(g_1g_2)x(g_1), x(g_1g_2)x(g_2))_v s(g_1)s(g_2)s(g_1g_2).$$

An advantage of this particular section is that $K_p = \mathbf{s}(\mathrm{SL}_2(\mathbb{Z}_p))$ is a subgroup in $G(\mathbb{Q}_p)$ if $p \neq 2$. If $p = 2$, we define

$$K_1(4) = \left\{ (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, 1) \in \mathrm{SL}_2(\mathbb{Z}_2) \times \{\pm 1\} : a \in 1 + 4\mathbb{Z}_2, c \in 4\mathbb{Z}_2 \right\}.$$

By Proposition 2.14 in [G2], $K_1(4)$ is a compact subgroup of $G(\mathbb{Q}_2)$.

A smooth representation of $G(\mathbb{Q}_v)$ is call *genuine* if μ_2 acts non-trivially. If p is an odd prime number, a smooth genuine representation of $G(\mathbb{Q}_p)$ is called *unramified* if it contains a vector fixed by K_p. A vector fixed by K_p is called a *spherical* vector.

If $p = 2$, a smooth genuine representation is called *tamely ramified* if it contains a vector fixed by $K_1(4)$. Unfortunately $\mathrm{SL}_2(\mathbb{Z}_2)$ does not split in $G(\mathbb{Q}_2)$ so we could not define spherical vectors in the same manner as those for odd primes. The objective of this paper is to motivate and define spherical vectors of genuine representations of $G(\mathbb{Q}_2)$.

We set up some notations for the later sections. For $u \in \mathbb{Q}_v$ and $t \in \mathbb{Q}_v^\times$, we define the following elements in $\mathrm{SL}_2(\mathbb{Q}_v)$:

$$\underline{x}(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \underline{y}(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \underline{w}(t) = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} \text{ and } \underline{h}(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

Let $x(u) = \mathbf{s}(\underline{x}(u))$, $y(u) = \mathbf{s}(\underline{y}(u))$, $w(t) = \mathbf{s}(\underline{w}(t))$ and $h(t) = \mathbf{s}(\underline{h}(t))$ in $G(\mathbb{Q}_v)$. Note that

$$h(t)h(s) = h(ts)(t, s)_v.$$

Let $N = \{x(u) : u \in \mathbb{Q}_v\}$, $\bar{N} = \{y(u) : u \in \mathbb{Q}_v\}$ and T be the subgroup of G generated by elements $h(t)$.

3. Hecke algebra at $p = 2$

We fix $p = 2$ throughout Sections 3 to 6. We will denote $G(\mathbb{Q}_2)$ by G and $K_1(4)$ by K_1. The objective of these sections is to classify genuine representations of G containing a non-zero vector fixed by K_1.

Let M be the center of G. It is a cyclic group of order 4 generated by $h(-1)$. (Note that $h(-1)h(-1) = (-1, -1)_2 = -1 \in \mu_2$.) Thus, a genuine central character γ is determined by its value on $h(-1)$, which is a fourth root of 1. Let K and K_0 be the open compact subgroups in G equal to the inverse images of $\mathrm{SL}_2(\mathbb{Z}_2)$ and

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_2) : c \in 4\mathbb{Z}_2 \right\}$$

respectively. Let $K(4) \subset K_1$ denote the principal congruence subgroup. It is the image under the section $\mathbf{s}$ of the subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$ consisting of matrices congruent to 1 modulo 4. We have $K \supset K_0 \supset K_1 \supset K(4)$ and $K_0 = M \times K_1$. We extend the central character γ to K_0, so that it is trivial on K_1. Given a smooth representation (π, V) of G, we denote

$$V^\gamma := \{v \in V : \pi(k_0)v = \gamma(k_0)v \text{ for all } k_0 \in K_0\}.$$

Let $\mathcal{R}(G,\gamma)$ denote the category of admissible smooth (necessarily genuine) representations V of G such that V^γ generates V as a G-module.

Next we define the corresponding Hecke algebra. Let $C_c(G)$ denote the set of locally constant, compactly supported functions on G. Let

$$H(\gamma) = \{f : C_c(G) : f(k_0 g k_0') = \overline{\gamma}(k_0) f(g) \overline{\gamma}(k_0') \text{ for all } k_0, k_0' \in K_0\}.$$

For $f_1, f_2 \in H(\gamma)$, we define

$$f_1 \cdot f_2(g_0) = \int_G f_1(g) f_2(g^{-1} g_0) dg = \int_G f_1(g_0 g) f_2(g^{-1}) dg$$

where dg is the Haar measure on G such that the measure of K_0 is 1. Then $H(\gamma)$ is a $\mathbb{C}$-algebra. For $f \in H(\gamma)$ and $v \in V$, we have

$$\pi(f)v = \int_G f(g)\pi(g) v dg \in V^\gamma.$$

In this way V^γ is a left $H(\gamma)$-module. Let $\mathcal{R}(H(\gamma))$ denote the category of finite dimensional left $H(\gamma)$-modules. We have a functor $A : \mathcal{R}(G,\gamma) \to \mathcal{R}(H(\gamma))$ given by $V \mapsto V^\gamma$. Since the group K_0 has a triangular decomposition

$$K_0 = (K_0 \cap \bar{N})(K_0 \cap T)(K_0 \cap N)$$

the functor A is an equivalence of categories. This follows, in essence, from [Ca], Corollary 3.3.6 (see also [Bo] and Theorem 4.2 in [BZ]).

Our immediate goal is to understand the structure of $H(\gamma)$. The character γ of the center M extends to a character γ of T which is trivial on $K_1 \cap T$ and $\gamma(h(2^n)) = 1$ for all $n \in \mathbb{Z}$. Let us abbreviate

$$\gamma(t) = \gamma(h(t)).$$

We define

$$\zeta = \frac{1 + \gamma(-1)}{\sqrt{2}}.$$

Note that ζ is a primitive 8-th root of 1. The character γ of T is invariant under conjugation by $w = w(1)$. We can now extend the character γ from T to the normalizer $N_G(T)$ by defining $\gamma(w) = \zeta$.

We define some functions in $H(\gamma)$. For g in $N_G(T)$ we set X_g to be the function supported on $K_0 g K_0$ such that

$$X_g(k_0 g k_0') = \overline{\gamma}(k_0)\overline{\gamma}(g)\overline{\gamma}(k_0')$$

for all $k_0, k_0' \in K_0$. Note that this definition depends only on the image of g in the affine Weyl group $W_a := N_G(T)/(T \cap K_0)$.

Proposition 1. *Functions X_g for g in W_a form a basis of $H(\gamma)$.*

Proof. We need first to determine the K_0-double cosets in G. This can be easily determined in $\mathrm{SL}_2(\mathbb{Q}_2)$ using the row-column reduction. In addition to $h(2^n)$ and $w(2^{-n})$ the double coset representatives are:

$$y(2), h(2^n)y(2), y(2)h(2^{-n}), y(2)w(2^{-n}), w(2^{-n})y(2) \text{ and } y(2)w(2^{-n})y(2)$$

where $n \geq 1$ in all cases. We claim that the Hecke algebra is not supported on these cosets.

Lemma 2. *The commutator of $x(2)$ and $y(2)$ modulo the principal congruence subgroup $K(4)$ is equal to $-1 \in \mu_2$.*

Proof. This can be easily checked using the multiplication rule. It also follows from applying Corollary 2.9 in [St] to the ring $A = \mathbb{Z}/4\mathbb{Z}$, □

Now we can easily finish the proof of proposition. Indeed if f is in $H(\gamma)$ then

$$f(y(2)) = f(y(2)x(2)) = -f(x(2)y(2)) = -f(y(2))$$

by the above lemma. This implies that f must vanish on $y(2)$. Other cases are dealt with in the same manner. □

Let $\ell : N_G(T) \to \mathbb{Z}$ be defined by $\ell(g) = \log_2(n)$ where n is the number of left (or right) K_0-cosets in the double coset $K_0 g K_0$. In other words, the volume of $K_0 g K_0$ is $2^{\ell(g)}$. For example, $w(2^{-1})$ normalizes K_0, so $\ell(w(2^{-1})) = 1$.

Proposition 3. *For every integer n we have $\ell(h(2^n)) = 2|n|$ and $\ell(w(2^{-n})) = 2|1-n|$. More precisely, we have the following decompositions of double co-sets:*

(i) If $n \geq 0$,

$$K_0 h(2^n) K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n}\mathbb{Z}} x(u) h(2^n) K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n}\mathbb{Z}} K_0 h(2^n) y(4u).$$

(ii) If $n \geq 1$,

$$K_0 h(2^{-n}) K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n}\mathbb{Z}} y(4u) h(2^{-n}) K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n}\mathbb{Z}} K_0 h(2^{-n}) x(u).$$

(iii) If $n \geq 0$,

$$K_0 w(2^n) K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n+2}\mathbb{Z}} x(u) w(2^n) K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n+2}\mathbb{Z}} K_0 w(2^n) x(u).$$

(iv) If $n \geq 1$,

$$K_0 w(2^{-n}) K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n-2}\mathbb{Z}} y(4u) w(2^{-n}) K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n-2}\mathbb{Z}} K_0 w(2^{-n}) y(4u).$$

Proof. This is an easy consequence of the decomposition $K_0 = (K_0 \cap \bar{N})(K_0 \cap T)(K_0 \cap N)$. Details are left to the reader. □

We record the following tautological lemma:

Lemma 4. *Let g_1 and g_2 be two elements in $N_G(T)$. If $\ell(g_1 g_2) = \ell(g_1) + \ell(g_2)$ then $X_{g_1} \cdot X_{g_2} = X_{g_1 g_2}$.* □

Let

$$\begin{cases} T_n = X_{h(2^n)} \\ U_n = X_{w(2^{-n})}. \end{cases}$$

Proposition 5. *Let $T_w = \sqrt{2}^{-1} U_0$. We have the following identities where m, n are any integers unless specified otherwise.*

(i) $(T_w + 1)(T_w - 2) = 0$.
(ii) $U_1 \cdot U_1 = 1$.
(iii) If $m, n \geq 0$, *or* $m, n \leq 0$ *then* $T_m \cdot T_n = T_{m+n}$.
(iv) $U_1 \cdot T_n = U_{n+1}$ *and* $T_n \cdot U_1 = U_{1-n}$.
(v) $U_1 \cdot U_n = T_{n-1}$ *and* $U_n \cdot U_1 = T_{1-n}$.

Proof. All statements except the first follow from Lemma 4. For (i) we need to show $T_w^2 = T_w \cdot T_w = T_w + 2$. Since T_w^2 is supported in K this is equivalent to $T_w^2(1) = 2$ and $T_w^2(w(1)) = T_w(w(1))$. Suppose $f_1, f_2 \in H(\gamma)$ where f_1 is supported on $K_0 r K_0 = \bigcup_{i=1}^s r_i K_0$ (disjointed union). Then

$$f_1 \cdot f_2(g) \quad = \quad \sum_{i=1}^{s} f_1(r_i) f_2(r_i^{-1} g).$$

We can apply this observation to $f_1 = f_2 = T_w$. Proposition 3 (the case $n = 0$ in (iii)) gives a decomposition of $K_0 w(1) K_0$ into single single cosets. Hence

$$T_w^2(g) = \sum_{u \bmod 4} T_w(x(u)w(1)) \cdot T_w(w(-1)x(-u)g).$$

If $g = 1$, this gives $T_w^2(1) = 4T_w(w(1)) \cdot T_w(w(-1))$. Since $T_w(w(1)) = 2^{-1/2}\overline{\zeta}$ and $T_w(w(-1)) = 2^{-1/2}\zeta$, we obtain that $T_w^2(1) = 2$. If $g = w(1)$, then

$$T_w^2(w(1)) = T_w(w(1)) \sum_{u \bmod 4} T_w(y(u)).$$

If $u = 0$ or 2 then $y(u)$ is not in $K_0 w(1) K_0$ and $T_w(y(u)) = 0$. If $u = \pm 1$, then $y(u) = x(u)w(-u)x(u)$ and we can rewrite

$$T_w^2(w(1)) = T_w(w(1))[T_w(w(1)) + T_w(w(-1))] = T_w(w(1)).$$

This proves (i). □

Here is the main result of this section.

Theorem 6. *The Hecke algebra $H(\gamma)$ is generated by T_w and U_1 as an abstract $\mathbb{C}$-algebra modulo the relations*
(a) $(T_w - 2)(T_w + 1) = 0$ *and*
(b) $U_1^2 = 1$.

Proof. Suppose H is the abstract algebra generated by $U_0 = \sqrt{2} T_w$ and U_1 modulo the relations (a) and (b). We have a natural homomorphism of $\mathbb{C}$-algebras $B : H \to H(\gamma)$. By Proposition 1, $H(\gamma)$ is spanned by T_n and U_n and by Proposition 5, these elements are generated by U_0 and U_1. This shows that B is surjective. It remains to show that B is injective. Suppose $h \in H$ is in the kernel of B. Since U_0 and U_1 satisfy quadratic relations, $h = \sum_i c_i u_i$ where $c_i \in \mathbb{C}$ and $u_i \in H$ is of the form $U_1 U_0 U_1 U_0 \ldots$ or $U_0 U_1 U_0 U_1 \ldots$. Since $U_0 U_1 = T_1$, $B(u_i)$ is either T_n, $T_n U_1 = U_{1-n}$, $U_1 T_n = U_{n+1}$, or $U_1 T_n U_1 = T_{-n}$. These

elements have disjointed supports as functions in $H(\gamma)$. Therefore $B(h) = \sum_i c_i B(u_i) = 0$ implies that $c_i = 0$ and $h = 0$. This proves that B is an injection and Theorem 6. $\square$

We now give two consequences of Theorem 6:

Proposition 7. *The element $Z := \frac{T_1}{2} + (\frac{T_1}{2})^{-1}$ belongs to the center of $H(\gamma)$.*

Proof. By Proposition 5, T_1 and U_1 generate $H(\gamma)$. Clearly Z commutes with T_1. It suffices to show that Z commutes with U_1. Since $T_1 = U_0 U_1$ we can use quadratic relations satisfied by U_0 and U_1 to write

$$2Z = U_0 U_1 + U_1 U_0 - 2^{1/2} U_1. \tag{2}$$

Hence Z commutes with U_1. This proves the proposition. $\square$

Proposition 8. *For $n \geq 0$, T_n is an invertible element in the algebra $H(\gamma)$.*

Proof. Note that the quadratic relations satisfied by U_0 and U_1 imply that U_0 and U_1 are invertible. Since $T_1 = U_0 U_1$, T_1 is also invertible. Hence $T_n = T_1^n$ is invertible. $\square$

Suppose (π, V) is a representation in $\mathcal{R}(G, \gamma)$. Let $(V_N)^\gamma = \{v \in V_N : \pi_{V_N}(t)v = \gamma(t)v$ for all $t \in K_0 \cap T\}$. The invertibility of T_n implies (see Lemma 4.7 in [Bo]):

Corollary 9. *Suppose (π, V) is a representation in $\mathcal{R}(G, \gamma)$. Then the canonical map $V^\gamma \to (V_N)^\gamma$ is a bijection. In particular V_N is nonzero, and V cannot be a supercuspidal representation.* $\square$

4. Local Shimura correspondence

Let $G' = \mathrm{PGL}_2(\mathbb{Q}_2)$. Let I be its Iwahori subgroup and let H' be its Iwarhori-Hecke algebra. Let T'_w and U'_1 denote the characteristic functions of

$$I \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} I \text{ and } I \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} I$$

respectively. Then H' is the abstract $\mathbb{C}$-algebra generated by T'_w and U'_1 satisfying the same relations as Theorem 6 (a) and (b) (see [Ma]). This gives the next corollary.

Corollary 10. *The Hecke algebras $H(\gamma)$ and H' are isomorphic $\mathbb{C}$-algebras.* $\square$

Let $\mathcal{R}(H')$ denote the category of finite dimensional representations of H'. Let $\mathcal{R}(G', I)$ denote the category of admissible smooth representations V of G' such that V^I generates V as a G'-module. By [Bo] and [BZ], the functor $V \mapsto V^I$ is an equivalence of categories from $\mathcal{R}(G', I)$ to $\mathcal{R}(H)$. The isomorphism in Corollary 10 establishes an equivalence of categories between $\mathcal{R}(H(\gamma))$ and $\mathcal{R}(H')$. Hence the following four categories are equivalent:

$$\mathcal{R}(G, \gamma) \simeq \mathcal{R}(H(\gamma)) \simeq \mathcal{R}(H') \simeq \mathcal{R}(G', I).$$

If V is a representation in $\mathcal{R}(G, \gamma)$, then we call the corresponding representation in $\mathcal{R}(G', I)$ the *local Shimura lift* of V. We denote it by $\mathrm{Sh}(V)$.

Proposition 11. *Let V be a representation in $\mathcal{R}(G, \gamma)$. Then the following statements are equivalent.*

(i) The local Shimura lift $\mathrm{Sh}(V)$ *is a spherical representation of* G'.
(ii) The action of T'_w *on* $\mathrm{Sh}(V)^I$ *has an eigenvalue* 2.
(iii) The action of T_w *on* V^γ *has an eigenvalue* 2.

Proof. The projection map to $G'(\mathbb{Z}_2)$-fixed vectors in $\mathrm{Sh}(V)$ is given by $\frac{1}{3}(T'_w+1)$, since T'_w+1 is the characteristic function of $G'(\mathbb{Z}_2)$ and the volume of $G'(\mathbb{Z}_2)$ is 3. It follows that a $G'(\mathbb{Z}_2)$-fixed vector is an eigenvector of T'_w with eigenvalue 2. This proves the equivalence of (i) and (ii). The equivalence of (ii) and (iii) follows from Corollary 10. □

The above proposition motivates the following definition.

Definition. Let V be a smooth representation of G. An eigenvector of T_w in V^γ with an eigenvalue 2 is called a γ-*spherical* vector. The representation is called a γ-*unramified* or γ-*spherical* representation if it contains a γ-spherical vector.

5. Pseudo-spherical representation of K at $p=2$

We retain the notations in Sections 3 and 4 where $p=2$. In the previous section we defined a representation V of G to be unramified if $V^\gamma \neq 0$ and T_w has an eigenvalue 2 on V^γ. In this section we shall reinterpret this condition in terms of representations of K. We shall see that K has only two irreducible representations E such that $E^\gamma \neq 0$. For both representations E^γ is one dimensional and they are distinguished by the action of T_w on E^γ. That eigenvalue can be either 2 or -1, so we shall use the eigenvalue to denote the representations by $V(2)$ and $V(-1)$. Their dimensions are 2 and 4, respectively. Thus, a representation of G is unramified if and only if it contains the two-dimensional K-type $V(2)$, which we may call a pseudo-spherical type.

If $E^\gamma \neq 0$ then, by Frobenius reciprocity, the K-type E is a summand of a six dimensional induced representation

$$I_K(\gamma) := \mathrm{Ind}_{K_0}^K \gamma = \{\phi : K \to \mathbb{C} : \phi(k_0 k) = \gamma(k_0)\phi(k) \text{ for all } k \in K, k_0 \in K_0\}.$$

Here the group K acts on it by right translation. We denote this action by π_R. Let $H_K(\gamma)$ denote the subalgebra of $H(\gamma)$ consisting of functions supported on K. We have the action of $H(\gamma)$ on $I_K(\gamma)^\gamma$, also denoted by π_R. By Proposition 1, $H_K(\gamma) = \mathbb{C}1 \oplus \mathbb{C}T_w$ and it is a commutative subalgebra. The algebra $H_K(\gamma)$ is anti-isomorphic to the algebra $H_K(\bar\gamma)$ via the map $f \mapsto \hat f$ where

$$\hat f(g) = f(g^{-1}).$$

For $f \in H_K(\bar\gamma)$ and $\phi \in I_K(\gamma)$, we set

$$(\pi_L(f)\phi)(g) = \int_K f(k)\phi(k^{-1}g)dk \quad \text{for all } g \in K.$$

This action commutes with the right action π_R of K on $I_K(\gamma)$ and

$$H_K(\bar\gamma) = \mathrm{End}_K(I_K(\gamma)).$$

Note that $I_K(\gamma)^\gamma = H(\bar\gamma)$. The actions π_L and π_R of $H(\bar\gamma)$ and $H(\gamma)$ on $I_K(\gamma)^\gamma = H(\bar\gamma)$ are related by $\pi_L(\hat f) = \pi_R(f)$.

We define the functions $F_{-1} := \frac{1}{3}(2-T_w)$ and $F_2 := \frac{1}{3}(T_w+1)$ in $H_K(\gamma)$. Then $\{F_{-1}, F_2\}$ is a basis of idempotents of $H_K(\gamma)$.

For $j = -1, 2$, let $V(j) = \pi_L(\hat{F}_j)I_K(\gamma)$. In other words $V(j)$ is the eigenspace of $\pi_L(\hat{T}_w)$ on $I_K(\gamma)$ corresponding to the eigenvalue j. Note that $\hat{F}_j \in V(j)$ and $\pi_R(T_w)\hat{F}_j = j\hat{F}_j$. In particular $\hat{F}_2$ is a γ-spherical vector.

Proposition 12. *(i) We have $I_K(\gamma) = V(-1) \oplus V(2)$ where each summand is an irreducible representation of K.*

(ii) We have $\dim V(-1) = 4$ and $\dim V(2) = 2$.

(iii) The K-submodule $V(2)$ contains a γ-spherical vector $\hat{F}_2$. The space of γ-spherical vectors is one dimensional.

(iv) The K-submodule $V(-1)$ does not have any γ-spherical vector.

Proof. Since $\dim \mathrm{End}(I_K(\gamma)) = 2$, both $V(-1)$ and $V(2)$ are irreducible K-modules. This proves (i).

In order to compute the dimensions of $V(-1)$ and $V(2)$ we need the following lemma.

Lemma 13. *The operator $\pi_L(\hat{T}_w)$ as an element in $\mathrm{End}_K(I_K(\gamma))$ has trace 0.*

Proof. For $g \in K$, let $\phi_g \in I_K(\gamma)$ such that ϕ_g is supported on K_0g and $\phi_g(k_0g) = \gamma(k_0)$. Let S be a set of representatives of $K_0\backslash K$, then $\{\phi_g : g \in S\}$ is a basis of $I_K(\gamma)$. In order to prove the lemma, it suffices to show that $(\pi_L(\hat{T}_w)\phi_g)(g) = 0$. Indeed, this shows that the matrix of $\pi_L(\hat{T}_w)$ in the basis ϕ_g has all diagonal entries equal 0. Note that $\pi_L(T_w)\phi_g$ is supported on $K_0w(1)K_0g$. If $(\pi_L(\hat{T}_w)\phi_g)(g) \neq 0$, then $g \in K_0w(1)K_0g$ and $1 \in K_0w(1)K_0$. Since $K_0 \neq K_0w(1)K_0$, this is a contradiction. The lemma is proved. □

We have $\dim V(2) + \dim V(-1) = \dim I_K(\gamma) = [K : K_0] = 6$. By the above lemma, $2\dim V(2) - \dim V(-1) = 0$. This implies $\dim V(-1) = 4$ and $\dim V(2) = 2$ and proves Proposition 12(ii). We have $I_K(\gamma)^\gamma = H_K(\bar{\gamma})$ and $\pi_R(F_j)I_K(\gamma) = \mathbb{C}\hat{F}_j$ for $j = -1, 2$. The vector $\hat{F}_2$ is γ-spherical while $\hat{F}_{-1}$ is not. This proves Parts (iii) and (iv). □

Theorem 14. *A smooth representation V of G with central character γ is γ-unramified if and only if there is a nontrivial K-module homomorphism $l : V(2) \to V$. A vector in V which is a scalar multiple of $l(\hat{F}_2)$ is a γ-spherical vector of V.*

Proof. A γ-spherical vector in V would generated a representation of K where every irreducible K-submodule is isomorphic to an irreducible submodule of $I_K(\gamma)$. Now the theorem follows from Proposition 12. □

6. Unramified principal series representations at $p = 2$

In this section, we continue to assume $p = 2$ and notations as in Sections 3 to 5. We will show that γ-unramified representations appear as submodules of principal series representations.

We recall the character γ of T in Section 3. Let $(\pi_s, I(\gamma, s))$ be the normalized induced principal series representation where $I(\gamma, s)$ is the set of smooth functions $\phi : G \to \mathbb{C}$

satisfying

$$\phi(\epsilon x(u)h(t)g) = \epsilon\gamma(t)|t|^{s+1}\phi(g)$$

for all $\epsilon \in \mu_2$, $u \in \mathbb{Q}_2$ and $t \in \mathbb{Q}_2^\times$. The group G acts by left translation $(\pi_s(g)\phi)(g') = \phi(g'g)$.

Proposition 15. *An irreducible γ-unramified representation V is isomorphic to a submodule of some $I(\gamma, s)$.*

Proof. By Corollary 9, $(V_N)^\gamma$ is nonzero. Hence there is a nontrivial T-homomorphism $V_N \to \gamma\nu^{s+1}$ for some $s \in \mathbb{C}$. Here ν is the character $\nu(\underline{h}(t)) = |t|$. By Frobenius reciprocity, there is a nontrivial map $V \to I(\gamma, s)$ which is an injection because V is irreducible. $\square$

We recall that $K(4)$ is the principal congruence subgroup in K_1. Restricting functions ϕ in $I(\gamma, s)$ to K gives a natural isomorphism of K-modules

$$l : I_K(\gamma) \to I(\gamma, s)^{K(4)}.$$

Theorem 16. *The K-types $V(2)$ and $V(-1)$ appear with multiplicity one in $I(\gamma, s)$. The space $I(\gamma, s)^\gamma$ is 2-dimensional. It is spanned by $l(\hat{F}_2)$ and $l(\hat{F}_{-1})$.* $\square$

We will describe a scalar multiple ϕ_j of $l(\hat{F}_j) \in V_j$ which is more convenient for later calculations. Let $d_2 = 1$ and $d_{-1} = -2$, and define ϕ_j be the unique vector in $I(\gamma, s)$ whose restriction to K is given by

$$\phi_j(k) = \begin{cases} d_j\gamma(k) & \text{if } k \in K_0, \\ 2^{-1/2}\zeta\gamma(k_0k_0') & \text{if } k = k_0w(1)k_0' \in K_0w(1)K_0, \\ 0 & \text{otherwise.} \end{cases}$$

We define an intertwining map $M(s) : I(\gamma, s) \to I(\gamma, -s)$ by

$$(M(s)\phi)(g) = \int_{\mathbb{Q}_2} \phi(w(1)x(u)g)du$$

where g is in G and du is the Haar measure on $\mathbb{Q}_2$ such that the measure of $\mathbb{Z}_2$ is 1.

Proposition 17. *We have*

$$M(s)\phi_2 = \frac{\zeta}{\sqrt{2}}\left(\frac{1 - \frac{1}{2}(2^{-2s})}{1 - 2^{-2s}}\right)\phi_2 \quad \textit{and} \quad M(s)\phi_{-1} = -\frac{\zeta}{2\sqrt{2}}\left(\frac{1 - 2(2^{-2s})}{1 - 2^{-2s}}\right)\phi_{-1}.$$

Proof. Since the vector ϕ_j is unique up to a scalar in $I_K(\gamma)$, $M(s)\phi_j = c\phi_j$ for some $c \in \mathbb{C}$. It remains to determine $c = d_j^{-1}M(s)\phi_j(1)$.

If $u \notin \mathbb{Z}_2$, then $w(1)x(u) = (-1, u)_2 \cdot x(-u^{-1})h(u^{-1})y(u^{-1})$. We write $u^{-1} = 2^m v$ where $v \in \mathbb{Z}_2^\times$ and $m \geq 1$. Recall that $\gamma(t) = \gamma(h(t))$. Then

$$\begin{aligned} M(s)\phi_j(1) &= \int_{\mathbb{Z}_2} \phi_j(w(1)x(u))du + \sum_{m=1}^{\infty} 2^{m-1}\int_{\mathbb{Z}_2^\times}(-1, 2^m v)_2\phi_j(h(2^m v)y(2^m v))d^\times v \\ &= 2^{-1/2}\zeta + \sum_{m=1}^{\infty} 2^{-ms-1}\int_{\mathbb{Z}_2^\times}(-1, v)_2\gamma(2^m v)\phi_j(y(2^m v))d^\times v \end{aligned}$$

where $d^\times v$ is the Haar measure of $\mathbb{Z}_2^\times$ with total measure 1. Now $\phi_j(y(2^m v)) = 0$ if $m = 1$ and it is equal to 1 if $m \geq 2$. Since $\gamma(2^m v) = \gamma(2^m)\gamma(v)(2^m, v)_2$ and $\gamma(2^m) = 1$, we can rewrite

$$\begin{aligned} M(s)\phi_j(1) &= 2^{-1/2}\zeta + d_j \sum_{m=2}^{\infty} 2^{-ms-1} \int_{\mathbb{Z}_2^\times} (2,v)_2^m (-1,v)_2 \gamma(v) d^\times v \\ &= 2^{-1/2}\zeta + d_j \sum_{m=2}^{\infty} 2^{-ms-1} \frac{1}{4} \sum_{v \in (\mathbb{Z}/8\mathbb{Z})^\times} (2,v)_2^m (-1,v)_2 \gamma(v). \end{aligned}$$

The sum $\sum_{v\in(\mathbb{Z}/8\mathbb{Z})^\times}$ on the right is zero if m is odd, and equals $\sqrt{2}\zeta$ if m is even. Finally adding up all the terms gives the constant c and the lemma. $\square$

Let $s_0 = \frac{1}{2}$ or $\frac{1}{2} + \frac{i\pi}{\log 2}$. From the above proposition, ϕ_{-1} lies in the kernel of $M(s_0)$ so $I(\gamma, s_0)$ is reducible. Indeed $I(\gamma, s_0)$ has a unique irreducible quotient which is an even Weil representation.

Definition. Let $s_0 = \frac{1}{2}$ or $\frac{1}{2} + \frac{i\pi}{\log 2}$. The kernel of $M(s_0)$ is called the *Steinberg* representation of $G(\mathbb{Q}_2)$. We shall denote this representation by $St(\epsilon)$ where $\epsilon = \pm 1$ such that $2^{s_0} = \epsilon\sqrt{2}$.

We claim that $St(\epsilon)$ is an irreducible representation of $G(\mathbb{Q}_2)$. Indeed by Section 6 in [LS], for every $s \in \mathbb{C}$, we have

$$I(\gamma, s)_N^{\mathrm{ss}} \cong \gamma|\cdot|^{s+1} \oplus \gamma|\cdot|^{-s+1} \tag{3}$$

where $I(\gamma, s)_N^{\mathrm{ss}}$ is the semi-simplification of $I(\gamma, s)_N$ as a T-module. Hence $I(\gamma, s)$ has at most length 2. The claim now follows because $St(\epsilon)$ is a proper submodule of $I(\gamma, s_0)$. Also see Section 7 of [Sa].

Corollary 18. *The even Weil representation contains the irreducible K-module $V(2)$. It is a γ-unramified representation. The Steinberg representation contains the irreducible K-module $V(-1)$.* $\square$

Proposition 19. *Let $Z = \frac{T_1}{2} + (\frac{T_1}{2})^{-1}$ be the central element in the Hecke algebra $H(\gamma)$ as in Proposition 7. Then $\pi_s(Z)$ acts on $I(\gamma, s)^\gamma$ as the scalar $2^s + 2^{-s}$.*

Proof. By Corollary 9, the natural projection of $I(\gamma, s)$ on $I(\gamma, s)_N$ gives an isomorphism of $I(\gamma, s)^\gamma$ and $I(\gamma, s)_N$. From the decomposition of $K_0 h(2) K_0$ into single K_0-cosets (Proposition 3(i)) it follows that the action of T_1 on $I(\gamma, s)^\gamma$ corresponds to the action of $4\cdot\pi_{s,N}(h(2))$ on $I(\gamma, s)_N$. By (3) the eigenvalues of $\frac{T_1}{2}$ are 2^s and 2^{-s}. This proves the proposition. $\square$

Corollary 20. *An irreducible γ-unramified representation is uniquely determined by the eigenvalue of the action of Z on its γ-spherical vector.*

Proof. Suppose the irreducible γ-unramified representation is a subquotient of both $I(\gamma, s)$ and $I(\gamma, s')$. Then by Proposition 19, $2^s + 2^{-s} = 2^{s'} + 2^{-s'}$ which implies $2^s = 2^{s'}$ or $2^s = 2^{-s'}$. By Proposition 17 both $I(\gamma, s)$ and $I(\gamma, -s)$ have the same irreducible γ-unramified subquotient. This proves the corollary. $\square$

Corollary 21. *The Steinberg representation $St(\epsilon)$ corresponds to the one dimensional representation of $H(\gamma)$ given by $T_w = -1$ and $U_1 = -\epsilon$.*

Proof. We know that $T_w = -1$ on $St(\epsilon)^\gamma$. It remains to compute the action of U_1. Since $St(\epsilon)$ is a subquotient of $I(\gamma, s_0)$ where $2^{s_0} = \epsilon\sqrt{2}$, the central element Z acts on $St(\epsilon)$ by the scalar $\epsilon(2^{1/2} + 2^{-1/2})$. By (2) we have $2^{1/2}Z = T_wU_1 + U_1T_w - U_1$. Hence $U_1 = -\epsilon$ as claimed. $\square$

Let V be an irreducible γ-unramified representation. By Proposition 15, we may assume that V is the unique γ-unramified subquotient of $I(\gamma, s)$ for some $s \in \mathbb{C}$. By Proposition 11, its local Shimura lift $V' = \mathrm{Sh}(V)$ is an unramified irreducible representation of $G' = \mathrm{PGL}_2(\mathbb{Q}_2)$. Let B' be the Borel subgroup of G'. We may realize V' as the unramified irreducible subquotient of the normalized induced principal series representation $(\pi'_s, I'(t))$ with trivial central character. Here $I'(t) = \mathrm{Ind}_{B'}^{G'}\omega^t$ (normalized induction) where ω is the character

$$\omega \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = |a_1/a_2|.$$

Theorem 22. *If V is the unique γ-unramified irreducible subquotient of $I(\gamma, s)$, then its local Shimura lift $\mathrm{Sh}(V)$ is the unique unramified irreducible subquotient of $I'(s)$.*

Proof. Assume that $\mathrm{Sh}(V)$ is a subquotient of $I'(t)$. By Proposition 19 the central operator Z in $H(\gamma)$ acts on $I(\gamma, s)^\gamma$ by the scalar $2^s + 2^{-s}$. The corresponding operator Z' in the algebra H' acts on $I'(t)$ by $2^t + 2^{-t}$. Thus, $2^s + 2^{-s} = 2^t + 2^{-t}$. Solving the equation gives $2^s = 2^t$ or $2^s = 2^{-t}$. Both $I'(t)$ and $I'(-t)$ have the same irreducible subquotients so we may set $s = t$. This proves the theorem. $\square$

Corollary 23. *The principal series representation $I(\gamma, s)$ is reducible if and only if $s = \frac{1}{2}$ or $\frac{1}{2} + \frac{i\pi}{\log 2}$.*

Proof. Let V be the γ-unramified irreducible subquotient of $I(\gamma, s)$. Let W be the unramified irreducible subquotient of $I'(s)$. Then $V = I(\gamma, s)$ if and only if $\dim V^\gamma = 2$. By Theorem 22, $\dim V^\gamma = \dim W^I$. Now $\dim W^I = 2$ if and only if $I'(s)$ is irreducible. Finally $I'(s)$ is irreducible if and only if $s \neq \frac{1}{2}$ and $\frac{1}{2} + \frac{i\pi}{\log 2}$. $\square$

7. Automorphic forms

In this section we first review a connection between automorphic forms and classical modular forms of half integral weight. This is mostly a well known material that can be found in Chapters 2 and 3 of [G2], and in [Wa]. We then transfer the action of the Hecke algebra $H(\gamma)$ to the setting of classical modular forms.

Let $\mathbb{A} = \prod_v \mathbb{Q}_v$ be the ring of adeles over $\mathbb{Q}$. We recall K_p, $s(g)$ and the cocycle σ_v defined in Section 2. Let $G(\mathbb{A}) = \mathrm{SL}_2(\mathbb{A}) \times \{\pm 1\}$ as a set. For $g_1 = (g_{1,v})$, $g_2 = (g_{2,v}) \in \mathrm{SL}_2(\mathbb{A})$ and $\epsilon_1, \epsilon_2 \in \{\pm 1\}$, the group law on $G(\mathbb{A})$ is given by

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2\sigma(g_1, g_2))$$

where $\sigma(g_1, g_2) = \prod_v \sigma_v(g_{1,v}, g_{2,v})$. Then $\mathrm{pr} : G(\mathbb{A}) \to \mathrm{SL}_2(\mathbb{A})$ given by $\mathrm{pr}(g, \epsilon) = g$ is a two-fold cover which splits over the subgroup $\mathrm{SL}_2(\mathbb{Q})$. Since $\mathrm{SL}_2(\mathbb{Q})$ is perfect this splitting is unique and it is given by $\mathbf{s}_{\mathbb{Q}} : \mathrm{SL}_2(\mathbb{Q}) \to G(\mathbb{A})$

$$\mathbf{s}_{\mathbb{Q}}(g) = (g, s_{\mathbb{A}}(g))$$

where $s_{\mathbb{A}}(g) = \prod_v s(g_v)$.

We also need a description of a maximal compact subgroup in $G(\mathbb{R})$. Let

$$\underline{k}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

for $-\pi < \theta \leq \pi$. Then $\underline{K}_\infty := \{\underline{k}(\theta) : -\pi < \theta \leq \pi\}$ is a maximal compact subgroup in $\mathrm{SL}_2(\mathbb{R})$. Let $K_\infty = \{k(\theta) : -2\pi < \theta \leq 2\pi\}$ where

$$k(\theta) = \begin{cases} (\underline{k}(\theta), 1) & \text{if } -\pi < \theta \leq \pi, \\ (\underline{k}(\theta), -1) & \text{if } -2\pi < \theta \leq -\pi \text{ or } \pi < \theta \leq 2\pi. \end{cases}$$

Then K_∞ is a maximal compact subgroup of $G(\mathbb{R})$ and $\mathrm{pr}(K_\infty) = \underline{K}_\infty$. If r is an odd integer, then $k(\theta) \mapsto e^{i\frac{r}{2}\theta}$ defines a genuine character of K_∞

Let $A_{r/2}(4)$ denote the set of functions φ in $L^2(\mathrm{SL}_2(\mathbb{Q})\backslash G(\mathbb{A}))$ satisfying the following properties:

1. $\varphi(gk_1) = \varphi(g)$ for all $k_1 \in K_1(4) \prod_{p \neq 2,\infty} K_p$,
2. $\varphi(gk_0) = \gamma(k_0)\varphi(g)$ for all $k_0 \in K_0$ in $G(\mathbb{Q}_2)$ where $\gamma(-1) = -i^r$,
3. $\varphi(gr(\theta)) = e^{i\frac{r}{2}\theta}\varphi(g)$,
4. as a function on $G(\mathbb{R})$, φ is smooth and satisfies $\triangle\varphi = -\frac{r}{4}\left(\frac{r}{4} - 1\right)\varphi$ where $\triangle$ is the Casimir operator and
5. φ is cuspidal, ie. $\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(x(u)g)du = 0$ for all $g \in G(\mathbb{A})$.

A basis of $A_{r/2}(4)$ arises from cuspidal automorphic representations $\pi = \otimes_v \pi_v$ of $G(\mathbb{A})$ such that π_∞ is a holomorphic discrete series representation with the lowest weight $r/2$, π_p is unramified for all $p \neq 2$, and π_2 contains a $K_1(4)$-fixed vectors. In particular, $\pi_2^\gamma \neq 0$ for some central character γ. Note that γ is determined by r. Indeed, since the local components of $\mathbf{s}_{\mathbb{Q}}(h(-1))$ for $v \neq \infty, 2$ are contained in K_p, $\varphi(1) = \varphi(\mathbf{s}_{\mathbb{Q}}(h(-1))) = \gamma(-1)e^{i\pi r/2}\varphi(1)$ and we get $\gamma(-1) = -i^r$.

Let $\mathcal{H}$ be the complex upper half plane. For $\underline{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $g = (\underline{g}, \epsilon) \in G(\mathbb{R})$ and $z \in \mathcal{H}$, we define

$$gz = \underline{g}z = \frac{az + b}{cz + d}.$$

We define a holomorphic function on $\mathcal{H}$ by

$$J(g, z) = J((\underline{g}, \epsilon), z) := \epsilon\,(cz + d)^{1/2}.$$

Here we choose $w^{1/2}$ such that $-\frac{\pi}{2} < \arg(w^{1/2}) \leq \frac{\pi}{2}$. We call $J(g, z)$ a *factor of automorphy*. By Lemma 3.3 in [G2], it satisfies $J(gg', z) = J(g, g'z)J(g', z)$ for any two g and g' in $G(\mathbb{R})$.

Define a congruence subgroup $\Gamma_0(4)$ by

$$\Gamma_0(4) := G(\mathbb{R}) \cap (\mathbf{s}_{\mathbb{Q}}(\mathrm{SL}_2(\mathbb{Q})) \cdot K_0(4) \cdot \prod_{p\neq 2} K_p).$$

Similarly, define $\Gamma_1(4) \subseteq \Gamma_0(4)$ by replacing $K_0(4)$ with $K_1(4)$. Let $S_{r/2}(\Gamma_0(4))$ and $S_{r/2}(\Gamma_1(4))$ be the spaces of classical modular forms of weight $r/2$. By page 183 in [Kb], $S_{r/2}(\Gamma_0(4)) = S_{r/2}(\Gamma_1(4))$. We will denote this space by $S_{r/2}(4)$.

By Proposition 3.1 in [G2], there is a bijection $Q : A_{r/2}(4) \to S_{r/2}(4)$ which we will recall below: Given $\varphi \in A_{r/2}(4)$, then

$$(Q\varphi)(z) = \varphi(g_\infty) J(g_\infty, i)^r$$

where $z = g_\infty i \in \mathcal{H}$. Conversely, given $f \in S_{r/2}(4)$. Let $g \in G(\mathbb{A})$. By Lemma 3.2 in [G2], $g = g_{\mathbb{Q}} g_\infty k$ for some $g_{\mathbb{Q}} \in \mathbf{s}_{\mathbb{Q}}(\mathrm{SL}_2(\mathbb{Q}))$, $g_\infty \in G(\mathbb{R})$ and $k \in K_1(4) \prod_{p\neq 2,\infty} K_p$. Then

$$(Q^{-1}f)(g) = f(g_\infty(i)) J(g_\infty, i)^{-r}.$$

Using the bijection Q, we define another bijection between the spaces of operators

$$\mathbf{q} : \mathrm{End}_{\mathbb{C}}(A_{r/2}(4)) \to \mathrm{End}_{\mathbb{C}}(S_{r/2}(4))$$

by $\mathbf{q}(L) = QLQ^{-1}$. Since the Hecke algebra $H(\gamma)$ defined in Section 3 acts on $A_{r/2}(4)$ it is of interest to reinterpret this action in terms of classical modular forms.

Proposition 24. *Let U_1 and T_1 be the operators in the local Hecke algebra $H(\gamma)$ where $\gamma(-1) = -i^r$. Recall that $\zeta = \frac{1-i^r}{\sqrt{2}}$. For $f(z) \in S_{r/2}(4)$, we have*

(i) $(\mathbf{q}(U_1)f)(z) = \overline{\zeta}\,(2z)^{-r/2} f\left(-\dfrac{1}{4z}\right)$ *and*

(ii) $(\mathbf{q}(T_1)f)(z) = 2^{-r/2} \displaystyle\sum_{u=0}^{3} f\left(\frac{z+u}{4}\right)$.

Proof. (i) Suppose $\varphi = Q^{-1}(f) \in A_{r/2}(4)$. For every place v, let $w_v = w(2^{-1})$ be the element in $G(\mathbb{Q}_v)$ defined in Section 2. By Proposition 3(iv),

$$(U_1\varphi)(g_\infty) = \int_{K_0 w_2 K_0} U_1(k)\varphi(g_\infty k)dk = U_1(w_2)\varphi(g_\infty w_2) = \overline{\zeta}\varphi(g_\infty w_2).$$

Next, consider $\underline{w}(2^{-1})$ in $\mathrm{SL}_2(\mathbb{Q})$. By (2.30) in [G2], $s_{\mathbb{Q}}(\underline{w}(2^{-1})) = \prod w_v$. Since φ is left $\mathrm{SL}_2(\mathbb{Q})$-invariant, and right K_p-invariant for $p \neq 2$,

$$\overline{\zeta}\varphi(g_\infty w_2) = \overline{\zeta}\varphi(s_{\mathbb{Q}}(\underline{w}(2^{-1}))^{-1} g_\infty w_2) = \overline{\zeta}\varphi(\left(\prod_{v\neq 2} w_v^{-1}\right) g_\infty) = \overline{\zeta}\varphi(w_\infty^{-1} g_\infty).$$

Applying Q to the above equation gives (i). Part (ii) is proved analogously. $\square$

8. Kohnen's plus space

Hecke eigenforms in $S_{r/2}(4)$ correspond to cuspidal automorphic representations π such that π_∞ is a discrete series representation of lowest weight $\frac{r}{2}$, π_p is unramified for all $p \neq 2$, and π_2 has $K_1(4)$-fixed vectors. In particular, $\pi_2^\gamma \neq 0$ for the central character $\gamma(-1) = -i^r$. If π_2 is a principal series representation then π_2^γ is 2-dimensional and therefore the corresponding Hecke eigenspace in $S_{r/2}(4)$ is also 2-dimensional. Kohnen's plus space is introduced to resolve this ambiguity. In terms of the space of automorphic functions $A_{r/2}(4)$, it is clear what to do. Decompose

$$A_{r/2}(4) = A^+_{r/2}(4) \oplus A^-_{r/2}(4)$$

where $A^+_{r/2}(4)$ is the eigenspace of the local Hecke operator T_w with the eigenvalue 2, while $A^-_{r/2}(4)$ is the eigenspace with the eigenvalue -1. Since a presence of the eigenvalue 2 for T_w acting on π_2 eliminates a possibility that π_2 is a Steinberg representation, we see that there is a one to one correspondence between Hecke eigenforms in $A^+_{r/2}(4)$ and cuspidal automorphic representations π (as above) such that π_2 is a γ-unramified representation. The classical Kohnen plus space is (essentially) $Q(A^+_{r/2}(4))$ as it will be explained in a moment. Niwa defines two operators $\mathbf{T}_4$ and $\mathbf{W}_4$ on $S_{r/2}(4)$ [Ni]:

$$(\mathbf{T}_4 f)(z) = \frac{1}{4}\sum_{u=0}^{3} f\left(\frac{z+u}{4}\right) \quad \text{and} \quad (\mathbf{W}_4 f)(z) = (-2iz)^{-r/2} f\left(-\frac{1}{4z}\right).$$

Note that $\mathbf{W}_4^2 = 1$. Let $\kappa = \frac{r-1}{2}$. Niwa shows that the operator $\mathbf{W} = (-1)^{\frac{(r^2-1)}{8}} 2^{1-\kappa}\mathbf{W}_4\mathbf{T}_4$ on $S_{r/2}(4)$ satisfies[1] the quadratic relation

$$(\mathbf{W}+1)(\mathbf{W}-2) = 0.$$

Kohnen defines $S^+_{r/2}(4)$ and $S^-_{r/2}(4)$ to be the eigenspaces of $\mathbf{W}$ on $S_{r/2}(4)$ of eigenvalues 2 and -1 respectively [Ko]. Proposition 24 says that

$$\begin{cases} \mathbf{q}(U_1) = (-1)^{\frac{r^2-1}{8}}\mathbf{W}_4 \\ \mathbf{q}(T_1) = 2^{\frac{3}{2}-\kappa}\mathbf{T}_4 \end{cases}$$

where the sign $(-1)^{\frac{r^2-1}{8}}$ is the quotient of $\overline{\zeta} = \frac{1+i^r}{\sqrt{2}}$ and $i^{\frac{r}{2}} = \left(\frac{1+i}{\sqrt{2}}\right)^r$. Since $T_w = \sqrt{2}^{-1} T_1 U_1$ it follows that $\mathbf{q}(T_w)$ and $\mathbf{W}$ are conjugates of each other by $\mathbf{W}_4$. Thus the Kohnen's plus space is simply a conjugate of our space:

$$Q(A^+_{r/2}(4)) = \mathbf{W}_4(S^+_{r/2}(4)).$$

Since $\mathbf{W}_4$ commutes with the classical Hecke operators $\mathbf{T}_{p^2}$ for $p \neq 2$, $Q(A^+_{r/2}(4))$ and $S^+_{r/2}(4)$ are isomorphic as $\mathbb{C}[\mathbf{T}_{3^2}, \mathbf{T}_{5^2}, \ldots]$-modules.

There is another description of $S^+_{r/2}(4)$ in terms of Fourier coefficients. It consists of the cusp forms whose n-th Fourier coefficients vanishes whenever $(-1)^\kappa n \equiv 2, 3 \pmod 4$.

[1] In Kohnen's paper [Ko], the operator is $\mathbf{T}_4\mathbf{W}_4$ acting on the right, ie $\mathbf{T}_4$ acts first and $\mathbf{W}_4$ follows.

Kohnen also defines a Hecke operator $\mathbf{T}_4^+$ which preserves $S^+_{r/2}(4)$ in the following way: For $f(z) = \sum_n a_n q^n \in S^+_{r/2}(4)$, we set $(\mathbf{T}_4^+ f)(z) = \sum_n b_n q^n$ where the sum is taken over integers $n > 0$ and $(-1)^\kappa n \equiv 0, 1 \pmod 4$, and

$$b_n = a_{4n} + \left(\frac{(-1)^\kappa n}{2}\right) 2^{\kappa-1} a_n + 2^{r-2} a_{n/4}.$$

Here $a_{n/4} = 0$ if n is not a multiple of 4. The large parenthesis denotes the Legendre symbol.

We can now formulate and prove our main global results.

Theorem 25. *There is a one to one correspondence between Hecke eigenforms f in $S^+_{r/2}(4)$ and irreducible cuspidal automorphic representations $\pi = \otimes_v \pi_v$ in $L^2(\mathrm{SL}_2(\mathbb{Q})\backslash G(\mathbb{A}))$ such that*

(i) π_∞ is the discrete series representation of $G(\mathbb{R})$ with the lowest weight $\frac{r}{2}$.
(ii) π_p is unramified for all odd primes p.
(iii) π_2 is γ-unramified where $\gamma(-1) = -i^r$.
(iv) If $\mathbf{T}_4^+ f = \lambda_2 f$, then a γ-spherical vector in π_2 is an eigenvector for $Z = \frac{T_1}{2} + (\frac{T_1}{2})^{-1}$ with the eigenvalue $2^{1-\frac{r}{2}}\lambda_2$.

Note that λ_2 determines the eigenvalue of Z on a γ-spherical vector which in turn determines π_2 uniquely by Corollary 20.

Proof. The first three statements are clear, since $Q^{-1}(\mathbf{W}_4 f)$ is a Hecke eigenform in $A^+_{r/2}(4)$ which is contained in a cuspidal automorphic representation π with these properties. It remains to show (iv). We need the following lemma.

Lemma 26. *Let f be in $S^+_{r/2}(4)$. Then $\mathbf{T}_4^+ f = 2^{\frac{r}{2}-1}\mathbf{q}(Z) f$.*

Proof. Recall that T_1 is invertible by Proposition 8. Hence, it suffices to show that

$$2^{2-r/2}\mathbf{q}(T_1)\mathbf{T}_4^+ = \mathbf{q}(T_1^2 + 4).$$

If $f(z) = \sum_{n=1}^\infty a_n q^n \in S_{r/2}(4)$ then, by Proposition 24, $(\mathbf{q}(T_1)f)(z) = 2^{2-r/2}\sum_{n=0}^\infty a_{4n} q^n$. Thus, if $f(z) \in S^+_{r/2}(4)$, then one computes

$$2^{2-r/2}(\mathbf{q}(T_1)\mathbf{T}_4^+ f)(z) = (\mathbf{q}(T_1^2+4)f)(z) = \sum_n (2^{4-r}a_{16n} + 4a_n)q^n.$$

This proves the lemma. □

Now we can finish the proof of Theorem 25. If $\mathbf{T}_4^+ f = \lambda_p f$ then Lemma 26 implies that $Q^{-1}(f)$ is an eigenform for Z with the eigenvalue $2^{1-\frac{r}{2}}\lambda_2$. Since $\mathbf{W}_4 = (-1)^{\frac{r^2-1}{8}}\mathbf{q}(U_1)$ and Z commutes with U_1, $Q^{-1}(\mathbf{W}_4 f)$ is also an eigenform for Z with the same eigenvalue. This completes the proof of Theorem 25. □

If f is a Hecke eigenform in $S^+_{r/2}(4)$ then by Theorem 1(ii) in [Ko] the corresponding Shimura lift $f' = \mathrm{Sh}(f)$ is a Hecke eigenform in $S_{r-1}(\mathrm{SL}_2(\mathbb{Z}))$. Recall that $G' = \mathrm{PGL}_2$. There is a one to one correspondence between Hecke eigenforms f' in $S_{r-1}(\mathrm{SL}_2(\mathbb{Z}))$ and

irreducible cuspidal automorphic representations $\pi' = \otimes_v \pi'_v$ in $L^2(G'(\mathbb{Q})\backslash G'(\mathbb{A}))$ such that π'_∞ is a discrete series representation with the lowest weight $r-1$ and π'_p is unramified for all primes p. See Proposition 3.1 in [G1]. We recall the local Shimura lift $\mathrm{Sh}(\pi_2)$ in Proposition 11 of a γ-unramified representation π_2 of $G(\mathbb{Q}_2)$. The following corollary gives a precise representation-theoretic description of the Shimura correspondence at the place $p=2$.

Corollary 27. *Let f be a Hecke eigenform in $S^+_{r/2}(4)$.*

(i) Let $\pi = \otimes_v \pi_v$ be the cuspidal automorphic representation corresponding to f in Theorem 25.
(ii) Let $\pi' = \otimes_v \pi'_v$ be the cuspidal automorphic representations of $L^2(G'(\mathbb{Q})\backslash G'(\mathbb{A}))$ corresponding to the Hecke eigenform $f' = \mathrm{Sh}(f)$ in $S_{r-1}(\mathrm{SL}_2(\mathbb{Z}))$.

Then $\mathrm{Sh}(\pi_2) = \pi'_2$.

Proof. If $\mathbf{T}^+_4 f = \lambda_2 f$ then by Theorem 1(ii) in [Ko], $\mathbf{T}_2 f' = \lambda_2 f'$ where $\mathbf{T}_2$ is the classical Hecke operator action on $S_{r-1}(\mathrm{SL}_2(\mathbb{Z}))$. By Proposition 5.2.1 in [G1] one checks that π'_2 is indeed isomorphic to $\mathrm{Sh}(\pi_2)$. □

Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ as in Theorem 25 and π' be the corresponding cuspidal automorphic representation of $G'(\mathbb{A})$ as in Corollary 27. By the Ramanujan conjecture, proved by Deligne, $\pi'_2 = \mathrm{Sh}(\pi_2)$ is a tempered irreducible unramified representation so $\pi'_2 = I'(s)$ for some $s \in i\mathbb{R}$. This implies that $\pi_2 = I(\gamma, s)$ by Theorem 22 and Corollary 23. In particular π_2^γ is an irreducible $H(\gamma)$-module of dimension 2. It corresponds under Q to a two dimensional subspace of $S_{r/2}(4)$ spanned by a line in $S^+_{r/2}(4)$ and a line in $S^-_{r/2}(4)$.

On the other hand, if $\pi_2 = St(\epsilon)$ is a Steinberg representation of $G(\mathbb{Q}_2)$ (see the definition before Corollary 18), then π corresponds under Q to an Hecke eigenform in $S^-_{r/2}(4)$. More precisely, we have the following theorem:

Theorem 28. *There is a one to one correspondence between Hecke eigenforms f in $S^-_{r/2}(4)$ such that $\mathbf{W}_4 f = -\epsilon(-1)^{\frac{r^2-1}{8}} f$, for some $\epsilon = \pm 1$, and irreducible cuspidal automorphic representations $\pi = \otimes_v \pi_v$ in $L^2(\mathrm{SL}_2(\mathbb{Q})\backslash G(\mathbb{A}))$ such that*

(i) π_∞ is the discrete series representation of $G(\mathbb{R})$ with the lowest weight $\frac{r}{2}$.
(ii) π_p is unramified for all odd primes p.
(iii) π_2 is the Steinberg representation $St(\epsilon)$.

Proof. Recall, by Corollary 21 that T_w and U_1 act on one-dimensional space $St(\epsilon)^\gamma$ by -1 and $-\epsilon$. The theorem now follows form Proposition 24 and the definition of $S^-_{r/2}(4)$. □

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Hung Yean Loke, Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543

E-mail address: `matlhy@nus.edu.sg`

Gordan Savin, Department of Mathematics, University of Utah, Salt Lake City, UT 84112

E-mail address: `savin@math.utah.edu`